Chaotic, staggered, and polarized dynamics in opinion forming: The contrarian effect

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We reconsider the no-tie-breaking two-state Galam contrarian model of opinion dynamics for update groups of size 3. While the initial model assumes a constant density of contrarians a for both opinions, the density now depends for each opinion on its global support. Proportionate contrarians are thus found to indeed preserve the main results of the former case. However, restricting the contrarian behavior to only the current collective majority makes the dynamics more complex with additional features. For a density $a < a_c = 1/9$ of one-sided contrarians, a chaotic basin is found in the 50-50 region separated from two majority-minority point attractors, one on each side. For $1/9 < a \lesssim 0.301$ only the chaotic basin survives. In the range $a > 0.301$ the chaotic basin disappears and the majority starts to alternate between the two opinions with a staggered flow toward two point attractors. We then study the effect of both decoupling the local update time sequence from the contrarian behavior activation and a smoothing of the majority rule. A status quo–driven bias for contrarian activation is also considered. Introduction of unsettled agents driven in the debate on a contrarian basis is shown only to shrink the chaotic basin. The model may shed light on recent apparent contradictory elections with on the one hand very close results as in the United States in 2000 and in Germany in 2002 and 2005, and on the other hand, a huge majority as in France in 2002.

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I. INTRODUCTION

In recent years the study of opinion dynamics has attracted a growing amount of work [1–12] making it a major current trend of sociophysics [13,14]. Most models consider two-state opinion agents combined with some local opinion update rule which implements the dynamics. They are found to lead to an opinion polarization of the whole population along one of the two competing opinions. A unifying frame was proposed to incorporate all these models [15]. Continuous opinion models yield a similar tendency [16,17].

More recently, the concept of the contrarian was introduced to account for some peculiar behavior of agents [18]. A contrarian is indistinguishable from others, i.e., its opinion evolves also by local rule updates. However, once it leaves the update group, it individually changes its opinion to the other one. The shift is independent of the opinion itself. A contrarian is not a permanent individual state. Each agent has a probability a to behave like a contrarian and $(1-a)$ to stick to its opinion. After each cycle of local updates, on average a proportion a of agents spontaneously shifts their opinion to the other one, while a proportion $(1-a)$ sticks to its current opinion. Given a fixed density of contrarians a, the associated opinion dynamics is then studied [18].

As intuitively expected, the existence of contrarians was shown to avoid total opinion polarization with the creation of stable attractors characterized by a stable coexistence between a large majority and a small minority. However, above some low density, they were found to produce an unexpected reversal of the dynamics, with the merger of the two attractors at the former separator, turning it into a unique stable attractor [18,19]. Accordingly, for whatever initial conditions the dynamics leads to an exact global balance between the two competing opinions. It thus offers a possible explanation of recently observed hung elections [18]. A hung election is a two-candidate run for which the result is very closely tied around 50% for each. A chaotic regime was also found in the description of investors in stock markets [20].

But in today's campaigns, polls are regularly publicized, making agents aware of which opinion is currently leading at the global level. It is therefore more natural to link the propensity to a contrarian behavior to the current level of global support for a given opinion. Accordingly, in this paper we relax the above constraints of fixed independent contrarian density a in two successive steps. First, we study the effect on the dynamics of making the density a for each opinion proportional to its current global support. Contrarians become proportionate contrarians. Second, we push the asymmetry further by restricting the contrarian behavior to only the current majority opinion, contrarians being one sided. While proportionate contrarians are found to preserve the mean features of the fixed density contrarian dynamics, a more complex situation including a chaotic regime is discovered for one-sided contrarians.

For a density $a < a_c = 1/9$, one-sided contrarians produce a chaotic basin located in the 50-50 region. The associated Lyapunov exponent is calculated. However, there still exist majority-minority coexistence point attractors located on each side of the chaotic basin. The initial conditions determine which regime will dominate, either the chaotic outcome around 50% for each opinion, or a point attractor with a well-defined majority. For $1/9 < a \lesssim 0.301$ only the chaotic basin survives. Further, in the range $a > 0.301$. the chaotic basin disappears and the majority starts to alternate between

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the two opinions with a staggered flow toward two majority-minority point attractors.

On this basis the effect of additional social parameters is studied. A constant shift for contrarian activation is considered. Decoupling of the local update time sequence from the contrarian behavior and a delay in accounting for a change of majority side are also introduced. Last, unsettled agents are included. They are found only to shrink the chaotic basin making the outcome very closely tied.

Given the above framework, the results may shed light on recent very unusual hung elections as in the 2000 U.S. presidential vote and in Germany for the 2005 elections. In both cases the outcomes were very close with almost identical support. At the same time, the model can also provide an explanation for the 2002 French presidential election where the winner obtained a huge majority around 80%.

At this stage it is worth stressing that we are able to embody these contradictory voting outcomes within a single frame. But at the same time, we are not able to decide beforehand which one will prevail. To overpass this difficulty would require collaboration with social scientists to estimate the actual type of contrarians involved in a given election.

The rest of the paper is organized as follows. In the next section we review the original Galam model of contrarians. Throughout the paper, the size of update local groups is kept equal to three agents. The contrarians are then made proportionate in Sec. III. Section IV considers one-sided contrarians and the corresponding complex dynamics topology. Unusual features are obtained. In Sec. V we study the effect of decoupling the local update time sequence from the contrarian behavior activation. A smoothing of the majority rule is also included. The effect of a status quo–driven bias for contrarian activation is investigated in Sec. VI. Section VII deals with the case of unsettled agents. The last section contains some discussion.

II. THE ORIGINAL GALAM MODEL: INDIVIDUAL CONTRARIANS

We start by recalling the Galam model of two-state opinion dynamics extended to the presence of contrarians [18]. It considers a population of N agents where at a time t, $N_A(t)$ persons support one opinion A and $\{N-N_A(t)\}$ persons support another competing opinion B. In terms of global proportions among the whole population, it yields $p_t=N_A(t)/N$ for A and $\{1-p_t\}$ for B. These values can be evaluated at any time using polls.

From an initial value p_t at time t, the dynamics is implemented in two steps. First is a neighborhood step where agents are distributed randomly among various size groups in which they update their respective individual opinions following the local initial majority [9]. In case of a tie in even size groups, either one opinion is adopted according to some probabilities [21]. The step is accounted for by a discrete time increment of +1 leading to a new proportion p_{t+1} of agents supporting opinion A. The second step is contrarian; each agent individually either shifts its respective opinion to the other one with a probability a, or preserves its current opinion with probability $(1-a)$. This second step yields an additional increment of time +1 and modifies p_{t+1} to another value p_{t+2}.

Throughout this paper we make explicit the calculations for the case of local groups with the unique value 3. The above rules thus yield for the first step $p_t \to p_{t+1}=P_m(p_t)$ with

$$p_{t+1} = P_m(p_t) \equiv p_t^3 + 3p_t^2(1-p_t), \tag{1}$$

and for the contrarian second step $p_{t+1} \to p_{t+2}=P_c(p_{t+1})$ where

$$p_{t+2} = P_c(p_{t+1}) \equiv (1-a)p_{t+1} + a[1-p_{t+1}]. \tag{2}$$

It should be stressed that before performing another cycle of opinion updates, agents are reshuffled [22]. Applying the above two-step cycle repeatedly n times results in a proportion p_{t+2n} of agents supporting A. All possible variations and extensions of the model can be included within a unifying frame [15].

At this stage we make a change of variable from p to d with $p=d+\frac{1}{2}$. It will appear to be more convenient for our investigation. A positive d makes A the majority opinion while a negative value grounds it as minority with a deficit $|d|$ of support with respect to B. In terms of the new variable d, $P_m(p_t)$ and $P_c(p_{t+1})$ become, respectively,

$$d_{t+1} = D_m(d_t) \equiv -2d_t^3 + \frac{3}{2}d_t \tag{3}$$

and

$$d_{t+2} = D_c(d_{t+1}) \equiv (1-2a)d_{t+1}, \tag{4}$$

which combine for one full cycle into the single equation $d_{t+2}=D_c[D_m(d_t)]$, which we denote by

$$d_{t+2} = D_2(d_t) = (1-2a)\left\{-2d_t^3 + \frac{3}{2}d_t\right\}, \tag{5}$$

where the index 2 of D_2 means one local rule followed by one contrarian step with a time interval of 2 for the dynamics. At this stage such a two-step split could appear artificial but it will become instrumental later on to extend the dynamics to cases where one cycle is built out of $(k-1)$ consecutive local updates followed by one contrarian effect. These cases will be denoted by $d_{t+k}=D_k(d_t)$ with k being the appropriate time interval to study the associated properties of the dynamics. In the original Galam work $k=2$.

For this last case the main results obtained from Eqs. (1), (2) or (3), (4) are twofold. At low concentration a of contrarian behavior, total polarization is prevented with two mixed attractors at which a majority and a small minority coexist. The threshold for A victory is at $p_v=\frac{1}{2}$ or $d_v=0$ which defines the separator of the dynamics. Furthermore, increasing a provokes a continuous phase transition at $a_c=\frac{1}{6}$ turning $p_v=\frac{1}{2}$ or $d_v=0$ into the unique and stable attractor of the dynamics. The final state is a perfect equality of both opinions at the collective level with ongoing individual opinion shifts [18].

III. MAKING CONTRARIAN BEHAVIOR CURRENT OPINION STATUS DEPENDENT: PROPORTIONATE CONTRARIANS

While the original model considers a constant and fixed proportion a of contrarians, it seems more realistic to make it depend on the current state of the system, in particular due to the existence of published polls. We thus suppose that if at some specific time t all agents are informed of the actual value of d_t, they react accordingly as contrarians with respect to d_t at time $(t+1)$ leading to d_{t+1}. But it thus becomes natural to make the contrarian behavior opinion dependent, not on the opinion itself but on its current level of support in the population.

To distinguish the associated contrarians from previous ones, we call them proportionate contrarians. Therefore, agents sharing opinion A react to p_t, i.e., d_t while agents sharing opinion B react to $(1-p_t)$, i.e., $-d_t$. We denote these rates, respectively, as $a(d)$ and $b(d)$. From symmetry we have $b(d)=a(-d)$ since opinions are time reversal symmetric.

For the time being, we keep our two-step cycle, which implies a regular and periodic publication of polls. Such a constraint will be relaxed in a later section. On this basis the proportionate contrarian density a becomes a function of time through the variable d_t. If no agent shares the opinion A, no one will react against it as a proportionate contrarian yielding the constraint $a(d=-1/2)=0$. On the other extreme at $d=1/2$, $a(d=1/2)=a_0$ where a_0 is the proportionate contrarian maximal value which satisfies $0 \leq a_0 \leq 1$.

Keeping in mind that now the proportionate contrarian density is not the same at a given time among agents sharing, respectively, opinions A and B, Eq. (4) becomes

$$d_{t+2} = D_2(d_t) = (1 - a_{t+1} - b_{t+1})d_{t+1} + \frac{b_{t+1} - a_{t+1}}{2}, \qquad (6)$$

where a_{t+1} and b_{t+1}, respectively, mean $a(d_{t+1})$ and $b(d_{t+1})$, and d_{t+1} is given by Eq. (3).

From Eq. (6) we can extract several properties of the associated dynamics. First we note that $d=0$ is a fixed point if and only if $a(d=0)=b(d=0)$. It means no agent is contrarian at perfect equality of opinions. Second we can evaluate its stability by studying the value of the associated eigenvalue λ with respect to 1. When $d=0$ is a fixed point, it is an attractor when $\lambda < 1$, which in turn implies the condition

$$6a(d=0) + a'(d=0) - b'(d=0) > 1, \qquad (7)$$

where the prime means a derivative with respect to d. Otherwise when $6a(d=0)+a'(d=0)-b'(d=0)<1$ $(\lambda > 1)$, the fixed point $d=0$ is a separator. A separator implies the existence of two attractors located one on each side at $d>0$ and $d<0$ with a stable coexistence of a majority and a minority.

Using the above results we can review few specific functional forms for the d dependence of a. We start with the linear dependence $a=a_0 p=a_0(1/2+d)$. For $a_0<1/5$, $d_v=0$ is a separator and the associated attractors are located at $d=\pm\sqrt{-1+5a_0}/2\sqrt{-1+a_0}$. When $a_0 \geq 1/5$, $d=0$ becomes the unique attractor of the dynamics. In other words, the former Galam result is recovered with proportionate contrarians stabilizing a perfect equality between both opinions once their density is larger than a critical value, here $1/5$ [18]. Results are shown in Fig. 1.

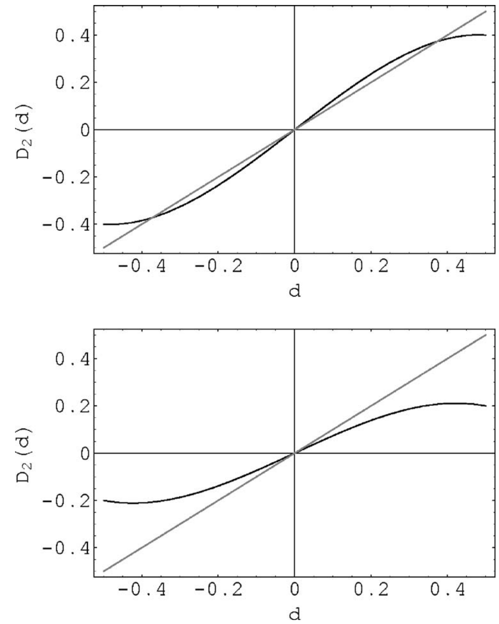

FIG. 1. Application D_2 given by Eq. (6) for the linear and symmetric case where $a(d)=a_0 p=a_0(1/2+d)$ and $b(d)=a(-d)$. On the top $a_0=0.1<1/5$ with two attractors at coexistence of a majority and a minority. On the bottom $a_0=0.3>1/5$ with one attractor at a perfect balance among both opinions.

Considering a power law form $a=a_0 p^{\gamma}=a_0(1/2+d)^{\gamma}$ we get the condition $a_0 \geq 2^{\gamma-1}/(3+2\gamma)$ to make $d=0$ the unique attractor. It is a separator with two mixed phase attractors when $a_0<2^{\gamma-1}/(3+2\gamma)$. Taking $\gamma=2$ yields a square dependence with $\frac{2}{7}$ for the critical value of a_0. More proportionate contrarians are needed $\left(a_0 \geq \frac{2}{7}=0.286\right)$ with respect to the linear case $\left(a_0>\frac{1}{5}=0.20\right)$ to produce the perfect balance of opinions. When $d=0$ is a separator, the two attractors are located at $d=\pm\frac{1}{2}\sqrt{1+(1/a_0)(1+\sqrt{1+8a_0^2})}$. In contrast, taking a square root dependence with $\gamma=1/2$ yields $1/4\sqrt{2} \simeq 0.177$ for the critical value of a_0, making fewer proportionate contrarians needed to get the opinion balance as the unique attractor of the dynamics. The former Galam model considers $a=cst$, giving a critical value of $\frac{1}{6} \simeq 0.167$ which is the lowest value we can get.

At this stage we can conclude that proportionate contrarians do not modify qualitatively the former constant contrarian density dynamics opinion.

IV. RESTRICTING CONTRARIAN BEHAVIOR TO THE CURRENT MAJORITY: ONE-SIDED CONTRARIANS

On the basis of the above results, we go back to the case of a constant density of contrarians, but now restricting the activation of the contrarian behavior to only the current majority opinion. It thus yields for $d>0$ the conditions $a(d>0)=a=cst$ (with $0 \leq a \leq 1$) and $b(d>0)=0$, while for $d<0$ the conditions are $a(d<0)=0$ and $b(d<0)=a=cst$,

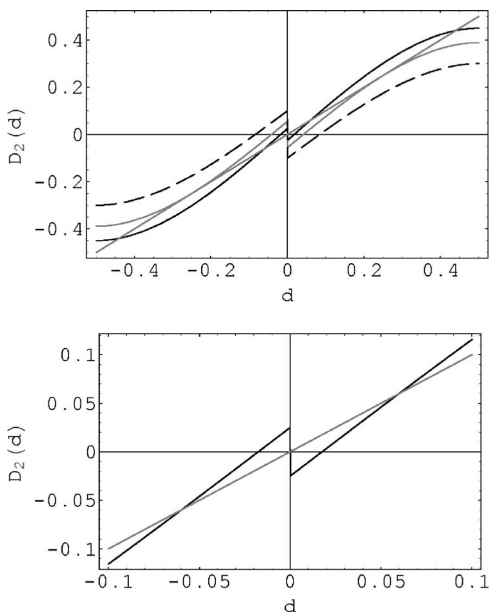

FIG. 2. Application D_2 given by Eq. (9). Top: for $a=0.05$ (plain line), $a_c=1/9$ (thin line), and 0.2 (dashed line). Bottom: for $a=0.05$, a zoom around $d=0$.

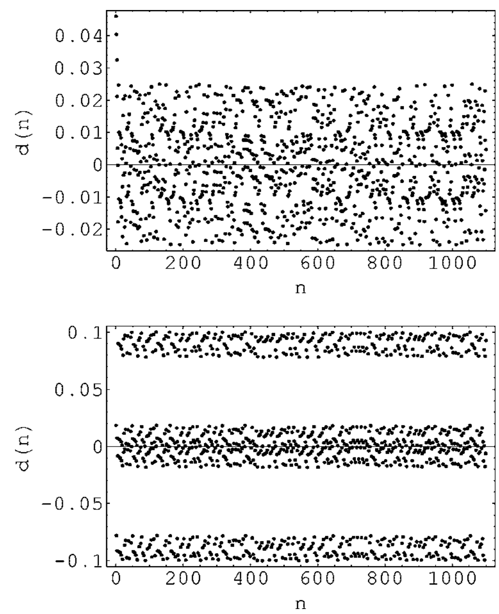

FIG. 3. Successive iterated points by the application D_2 given by Eq. (9). Notation here: $d(n+1)=D_2[d(n)]$. Top: $a=0.05$ with the initial value $d(0)=0.05$. Bottom: $a=0.2$ with the initial value $d(0)=0.1$.

where we have assumed symmetric conditions for both opinions. It is worth noting that this symmetry induces at $d=0$ the condition $a(d=0)=b(d=0)=a/2$ both preserving the same total density a of contrarians and recovering here the former original individual contrarian case. We call these contrarians one-sided contrarians. The associated rule update is written

$$d_{t+2}=D_{o-s}(d_{t+1}) \equiv (1-a)d_{t+1}-\frac{a}{2}\,\mathrm{sgn}[d_{t+1}], \tag{8}$$

where $\mathrm{sgn}[x]=1$ if $x>0$, -1 if $x<0$ and 0 if $x=0$.

Using Eq. (3) for d_{t+1} and noting that $\mathrm{sgn}[d_{t+1}]=\mathrm{sgn}[d_t]$, Eq. (6) is written

$$d_{t+2}=D_2(d_t)=(1-a)\left(\frac{3}{2}d_t-2d_t^3\right)-\frac{a}{2}\,\mathrm{sgn}[d_t]. \tag{9}$$

At second order in d_t it yields

$$D_2(d_t)\simeq \lambda_m d_t-\frac{a}{2}\,\mathrm{sgn}[d_t], \tag{10}$$

where $\lambda_m=\frac{3}{2}(1-a)$ is the maximal slope of the application D_2.

We can now study the associated fixed points and their stability. We first note that the point $d=0$ is a singular fixed point. Then, the application being symmetric, we restrict the study to $d>0$. According to Eq. (9), fixed points exist if $a\leq a_c=\frac{1}{9}\simeq 0.11$ and their values are

$$d_{F\pm}=\frac{1}{4}\left(1\pm\sqrt{\frac{1-9a}{1-a}}\right). \tag{11}$$

When they exist, d_{F-} is unstable and d_{F+} stable (see Fig. 2). From Eq. (10) it gives

$$d_{F-}\simeq\frac{a}{2(\lambda_m-1)}, \tag{12}$$

with the eigenvalue $\lambda\simeq\lambda_m>1$ ($\lambda_m>1$ for $a\leq a_c$).

Equation (10) exhibits clearly condition for expansion when $\lambda_m>1$, and folding up by the discontinuity at the origin $d=0$. Thus, to be chaotic, the application D_2 has to satisfy two conditions; first, the expansion condition and second, the perpetuity of the chaotic basin [23,24]. If the application D_2 is chaotic, the interval of successive iterated points after the transients is called Ω_l. Here $\Omega_l=]-a/2;a/2[$ (see Fig. 2).

First condition: Expansion. Inside the interval Ω_l the application D_2 possesses a slope λ, such as $\lambda_m>\lambda>D_2'(a/2)$, where D_2' denotes the derivative of D_2 relating to d. This implies that necessarily $\lambda_m>1$, i.e., $a<1/3$. Furthermore, $D_2'(a/2)=1$ for $a\simeq 0.27$. According to this unique condition, D_2 loses its chaotic nature for a including the two last values.

Second condition: Perpetuity of the chaotic basin, i.e., $D_2(\Omega_l)\subset\Omega_l$. If there are no fixed points, this condition is automatically satisfied. If fixed points exist, i.e., for $a\leq a_c$, they must be outside the interval $]-a/2;a/2[$, i.e., $d_{F-}>a/2$. According to Eq. (12) this implies $\lambda_m<2$. This condition is always satisfied here for a group of size 3 while it is not the case for larger update groups from size 5. Nevertheless, to observe chaotic behavior the initial condition must satisfy $|d(t=0)|<d_{F-}$. Otherwise, the successive iterated points go to a stable fixed point at $\pm d_{F+}$.

A chaotic behavior is numerically observed until $a\simeq 0.301$. Illustrations are shown in Fig. 3 for $a=0.2$ and 0.05 (with an appropriate initial condition in the latter case).

Figure 4 shows the sensitivity to initial conditions and permits the evaluation of the Lyapunov exponent λ_{Lyap}.

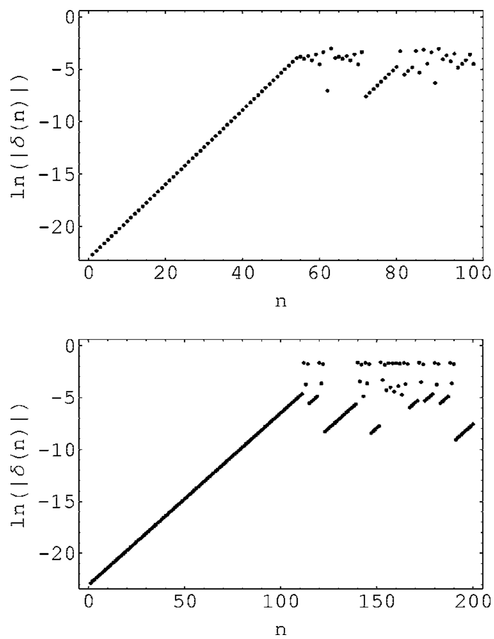

FIG. 4. Error growth and Lyapunov exponent of the application D_2 given by Eq. (9) in semilogarithmic plot. Notation here: $d(n+1)=D_2[d(n)]$. The initial error is $\delta(0)=10^{-10}$, taken after transients. Top: $a=0.05$. From the graph the Lyapunov exponent is $\lambda_{Lyap}\simeq 0.353$ while $\ln(\lambda_m)\simeq 0.354$. Bottom: $a=0.2$. From the graph the Lyapunov exponent is $\lambda_{Lyap}\simeq 0.166$ while $\ln(\lambda_m)\simeq 0.182$.

The initial difference $\delta(0)$ grows exponentially, $\delta(n)\simeq\delta(0)e^{n\lambda_{Lyap}}$, until saturation at the typical size of the interval Ω_l. The Lyapunov exponent is positive with $\lambda_{Lyap}\simeq\ln(\lambda_m)$ as seen from Eq. (10).

The notation adopted in these figures is to consider the nth iteration of the application D_2 since the beginning of the electoral campaign as the equivalent time n. Thus $d_{t_0+2n}\equiv d(n)$, where t_0 is the time at the beginning of the electoral campaign. In other words, $d(n+1)=D_2[d(n)]$.

At this stage we can make the following comments.

(1) The discontinuity of the application D_2 at $d=0$ is not the unique origin of its chaotic nature. Indeed, we can transform D_2 to be continuous and derivable, e.g., via $a\rightarrow a(d)=a_m(1-e^{-|d|/\alpha})$. If $a_m<0.301$ and $\alpha\ll 1$ then this application exhibits a chaotic behavior. Let us recall that sudden variation is the source of the chaotic nature of the application because it permits the folding up of an expansive application (if $\lambda_m>1$).

(2) When $a>0.311$ the application has stable fixed points of doubling period. They are obtained from $D_2(d)=-d$.

(3) Increasing a, the first separation for $d>0$ inside the interval Ω_l in two intervals occurs at $a\simeq 0.056$. This result can be retrieved considering the doubling iterated application $D_2^{(2)}=D_2\circ D_2$, i.e., $D_2^{(2)}(d)=D_2(D_2(d))$. Indeed, this occurs when $\lim_{d\rightarrow 0^-}D_2^{(2)}(d)>d^{(2)*}$, i.e., $D_2(a/2)>d^{(2)*}$, where $d^{(2)*}$ is a fixed point of the doubling period where $d^{(2)*}$ is obtained via $D_2(d^{(2)*})=-d^{(2)*}$. From Eq. (10) we retrieve $a\simeq 1-2\sqrt{2}/3\simeq 0.057$. The extension of successive iterated points after transients, Ω_l, is then, for positive values, $]0;D_2(a/2)[\cup]|D_2^{(2)}(a/2)|;a/2[$.

(4) Starting for instance from opinion A being initially the majority, i.e., $d>0$, we evaluate d_{ch}, such that $D_2(d_{ch})=0$.

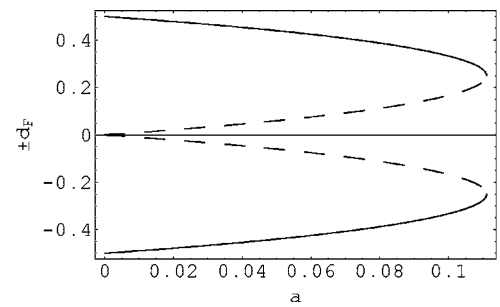

FIG. 5. Separators $\pm d_{F-}$ of chaotic basin (dashed line) and point attractors $\pm d_{F+}$ (plain line), given by Eq. (11) as functions of density a. $a\leq a_c=1/9$.

Thus, if $d<d_{ch}$ then $D_2(d)<0$ and the majority side will change to the opposite side; and reciprocally if $d>d_{ch}$ then $D_2(d)>0$ and the majority side will keep the same side (see Fig. 2). From Eq. (10)

$$d_{ch}\simeq\frac{a}{2\lambda_m}. \tag{13}$$

Since successive iterated points are contained in the interval $\Omega_l=]-a/2;a/2[$ and $D_2(a/2)<d_{ch}$ with $\lambda_m\leq 1.5$, we deduce that $d>0$ cannot remain positive more than twice before turning negative. Note that at second order on d, $D_2(a/2)>d_{ch}$ yields a second-order equation whose solution is $\lambda_m>(1+\sqrt{5})/2\simeq 1.62$, the golden number.

To summarize, for $a<a_c=\frac{1}{9}$ this model provides the coexistence of three radically different basins. A chaotic one located around $d=0$ delimited by separators $\pm d_{F-}$ and two others at the extremes with each one a point attractor $\pm d_{F+}$. Figure 5 shows positions of separators and point attractors as functions of a. For $1/9<a\lesssim 0.301$ only the chaotic basin survives. For $a\gtrsim 0.301$ the opinion forming staggers toward two stable fixed points of the doubling period. Most interesting effects are obtained for a density of one-sided contrarians such that $a<a_c=\frac{1}{9}$. Indeed, in this case the initial value $d(0)$ at the beginning of the electoral campaign determines which one basin is selected. If $|d(0)|<d_{F-}$ the intention vote dynamics is chaotic but with a result around $d=0$ such that $d\in]-a/2;a/2[$. If $|d(0)|>d_{F-}$, the issue is certain but with a result at the extremes. This could account for contradictory electoral outcomes whose results are either around $p=50\%$ or with a huge majority at $p\simeq 80\%$ as in the 2002 French presidential election.

V. VARYING THE TIME SCALES OF COLLECTIVE INFORMATION AND LOCAL UPDATES

After having considered two-step processes, we now study the opinion dynamics driven by a k-step process where the individual activation of the one-sided contrarian step occurs after $k-1$ repeated steps of local majority rule update. It accounts for the fact that polls are not made public every single day during a campaign while on the contrary people keep on discussing all the time.

Now $d_{t+k}=D_{o-s}(d_{t+k-1})$ with $d_{t+k-1}=D_m^{(k-1)}(d_t)$ where $D_m^{(k-1)}=D_m\circ D_m\circ\cdots\circ D_m$, i.e., $k-1$ iterations of D_m. In previous sections, k was equal to 2. Accordingly one-sided contrarians consider the collective information with a delay acting at time $t+k-1$ while considering information at time t. However, they could as well consider the collective information without delay, just after the last update inside groups at time $t+k-1$. Indeed, they act according to $\mathrm{sgn}[d]$ (see Sec. IV) and $\mathrm{sgn}[d_t]=\mathrm{sgn}[d_{t+1}]=\cdots=\mathrm{sgn}[d_{t+k-1}]$.

Moreover d_{t+k-1} increases very quickly from the origin $d=0$, causing contrarian behavior to have little effect. For instance $k=10$ makes the slope at the origin at $(\frac{3}{2})^9 \simeq 38$. So we have to slow down the dynamics driven by the update rule inside groups in order to study the varying time scale of collective information. It is done quite naturally by assuming that not every agent eventually changes its opinion to follow the local majority within each cycle of local updates, turning Eq. (1) to

$$p_{t+1} = P_{m,w}(p_t) \equiv w[3p_t^2 - 2p_t^3] + (1-w)p_t, \qquad (14)$$

where w denotes the propensity of an agent to be convinced by majority rule with $0 \leq w \leq 1$. Using the d variable gives

$$d_{t+1} = D_{m,w}(d_t) \equiv (1 + w/2)d_t - 2wd_t^3. \qquad (15)$$

Now $k-1$ iterations of $D_{m,w}$ yield a slower slope at the origin, e.g., for $w=0.1$ and $k=10$ it is $(1+w/2)^9 \simeq 1.5$.

Let a be the density of contrarians and w the propensity of an agent to be convinced by majority rule. The new intention vote dynamics of the k-step process, generated by the one-sided contrarian step occurring after $k-1$ repeated local majority rules, is written as

$$d_{t+k} = D_k(d_t) = D_{os}[D_{m,w}^{(k-1)}(d_t)], \qquad (16)$$

where D_{os} and $D_{m,w}$ are respectively given by Eqs. (8) and (15). This yields at second order on d_t Eq. (10), but now with the slope $\lambda_m=(1-a)(1+w/2)^{k-1}$.

To exhibit a chaotic behavior the condition for expansion $\lambda_m > 1$ gives now

$$k-1 > \frac{-\ln(1-a)}{\ln(1+w/2)}. \qquad (17)$$

If $a,w \ll 1$ and are of the same order, then $k-1 > 2a/w$, e.g., for $a=w \ll 1$, $k>3$. Numerically this can be satisfied until $a=w \leq 0.4$ (see Fig. 6).

With respect of the perpetuity of the chaotic basin, the unstable fixed points [Eq. (12)] are $\pm d_{F-} \simeq \pm a/2(\lambda_m-1)$, if they exist. This implies $\lambda_m > 1$ and $\lambda_m \leq 1+a$ to have $d_{F-} \leq 1/2$, so $k-1 > [\ln(\frac{1+a}{1-a})]/\ln(1+w/2)$. If $a,w \ll 1$ and are on the same order, then $k-1 > 4a/w$, e.g., for $a=w \ll 1$, $k>5$. Nevertheless, the second-order approximation is not proper as soon as the inequality $d_{F-} \leq 1/2$ no longer satisfies $|d| \ll 1$.

To check if successive iterated points cannot escape the chaotic basin, i.e., if $d_{F-} > a/2$, from Eq. (12) we need to have $\lambda_m < 2$, i.e.,

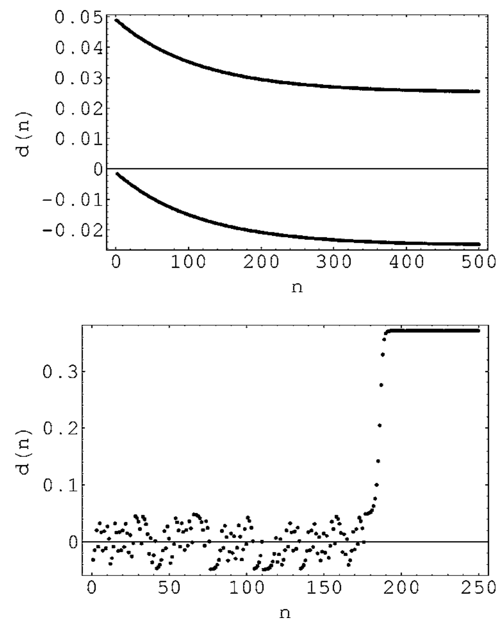

FIG. 6. Successive iterated points by the application D_k given by Eq. (16). Notation here: $d(n+1)=D_k[d(n)]$. Top: $a=w=0.1$, $k=3$, and an initial value $d(0)=0.1$. The application is not chaotic. $\lambda_m \simeq 0.99 < 1$. Bottom: $a=w=0.1$, $k=18$, and an initial value $d(0)=-0.04$. The application is not chaotic because it escapes from the previous chaotic basin. $\lambda_m \simeq 2.06 > 2$.

$$k-1 < \frac{\ln(2) - \ln(1-a)}{\ln(1+w/2)}. \qquad (18)$$

If $a,b \ll 1$ and are on the same order, then $k-1 < [2\ln(2) + 2a]/w$, e.g., for $a=w=0.1$, $k<16.9$ while numerically, $k<18$ (see Fig. 6). As in Sec. IV, the initial value has to satisfy $|d(t=0)| < d_{F-}$. However, in contrast to the previous section, here successive iterated points can escape the previous chaotic basin.

In addition, for $k>2$, this k process with $k-1$ repeated steps of local majority rule updates increases naturally the majority side persistence before changing compared to the previous model.

VI. STATUS QUO–DRIVEN BIAS

Up to now both opinions were perfectly symmetric. However, while dealing with political opinion dynamics in view of an election a difference should be made between the opinion supporting the current political party in power and the one supporting the challenging party. The contrarian behavior should be a little more active against the former winner opinion, thus creating a bias in favor of the challenging opinion [9].

Assuming B is the former winner opinion and denoting s the bias in favor of the challenging opinion, Eqs. (9) and (10) should be rewritten as

$$d_{t+2} = D_2(d_t) = (1-a)\left(\frac{3}{2}d_t - 2d_t^3\right) - \frac{a}{2}\,\mathrm{sgn}[d_t - s], \qquad (19)$$

and $D_2(d_t) \simeq \lambda_m d_t - (a/2)\mathrm{sgn}[d_t - s]$ where $\lambda_m = \frac{3}{2}(1-a)$.

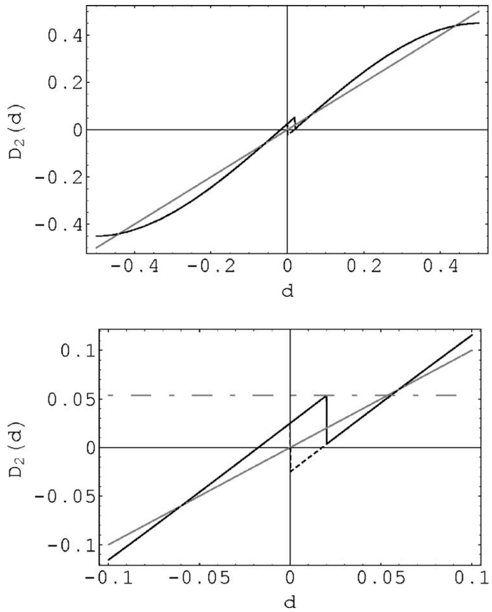

FIG. 7. Application D_2 given by Eq. (19) with a status quo–driven bias for $a=0.05$ and $s=0.02$ (plain line). The dashed line is the application without bias. At the right side, a zoom of this application around $d=0$.

In the case $a \leq a_c = 1/9$ for which fixed points exist, Fig. 7 ($a=0.05$ and $s=0.02$) shows that for $|s| < d_{F-}$, the bias does not modify the position of the fixed points. But now, depending on the ratio of s to a, the successive iterated points could escape from the previous chaotic basin, and thus reach a point attractor. The value s, at a fixed density a, for which this new phenomenon occurs, is obtained when $\mathrm{Sup}[\lim_{d_t \to s^-} D_2(d_t); \lim_{d_t \to s^+} |D_2(d_t)|] > d_{F-}$.

Thus, at first order in s, $(a/2 + \lambda_m |s|) > d_{F-}$ and at second order in d_t, Eq. (12) yields

$$|s| > s_c \simeq a \frac{2-\lambda_m}{2\lambda_m(\lambda_m - 1)}, \qquad (20)$$

where here $1 < \lambda_m < \frac{3}{2}$. For instance $a=0.05$ gives $s_c \simeq 2.37\%$ for the exact value $s_c \simeq 2.44\%$.

Evaluating the values s, at a fixed density a, for which successive iterated points d_t have the same sign, we find $|s| > d_{ch}$, where d_{ch} is defined as $D_2(d_{ch})=0$ for the application without bias (see Fig. 7). From Eq. (13)

$$|s| > a \frac{1}{2\lambda_m}. \qquad (21)$$

For instance $a=0.05$ yields $|s| > 1.75\%$ for the exact value $|s| > 1.76\%$. It is worth noting that for s sufficiently large the application is no longer chaotic. For instance, with $a=s=0.2$, the application has a periodic attractor of period 13.

Including a bias in favor of one opinion provides two main effects. First, after transients, the majority side (the favored opinion) before changing is more persistent than without bias. Second, for given density a of one-sided contrarians, it is possible to escape the chaotic basin without bias to reach the point attractor of the favored opinion.

VII. UNSETTLED PEOPLE AND CONTRARIAN ATTRACTION

We now go back to a population where agents sharing either opinion evolve by local majority rule updates only without contrarians [9]. But we also consider another population of agents who do not take part in the public debate. They are unsettled and hold no opinion. However, they are gradually driven in the public debate on a contrarian basis. At a constant rate u with $0 \leq u \leq 1$ they move to the opinion-holding population starting with an opinion opposite to the current majority. Once they adopt an opinion they become identical to other opinion-sharing agents, i.e., they evolve by local majority rules.

At time t the number of persons sharing opinion A, opinion B, and no opinion are denoted, respectively, $N_A(t)$, $N_B(t)$, and $N_U(t)$ with $N_O(t) = N_A(t) + N_B(t)$ and $N_O(t) + N_U(t) = N$ where N is the total number of agents of both populations. Associated probabilities are

$$p_t = \frac{N_A(t)}{N_O(t)} \quad \text{and} \quad \frac{N_B(t)}{N_O(t)} = 1 - p_t. \qquad (22)$$

We still have a two-step process. The first one is unchanged with $p_{t+1} = P_m(p_t)$ while the second one is produced by the contrarian unsettled agents partially joining the debate. Note that now the application $p_t \to p_{t+2}$ is no longer stationary. To account for the shrinking dynamics of unsettled agents we denote by n the time that corresponds to the nth iteration, i.e., $d_{t_0+2n} \equiv d(n)$ where t_0 is the time at the beginning of the campaign. Thus, writing $d(n+1) = D_{2,n}[d(n)]$, it gives

$$d(n+1) = D_{2,n}[d(n)] = \frac{\frac{3}{2}d(n) - 2d(n)^3}{1 + a(n)} - \frac{a(n)}{2[1 + a(n)]}\,\mathrm{sgn}[d(n)], \qquad (23)$$

with

$$a(n) = u \frac{N_U(n)}{N_O(n)} = u \frac{(1-u)^n}{R - (1-u)^n}, \qquad (24)$$

where $R = N/N_U(0) \geq 1$. For $u \ll 1$, on the first order in $uN_U(n)/N_O(n)$ Eqs. (9) and (10) are unchanged with now instead of a an effective time-dependent contrarian density: $a \to a(n)$.

In the limit $u \ll 1$ from Eq. (12) we can write $d_{n,F-} \simeq a(n)$ with $\Omega_{l,n}$ defined for $0 < |d| < a(n)/2$. With $\lim_{n\to\infty} a(n) = 0$ the interval $\Omega_{l,n}$ shrinks to 0. However, this model prohibits $n \to \infty$ since the number of unsettled persons mobilized at each iteration is an integer and not a real one, although the nth iteration, the number $uN_U(0)(1-u)^n$, is assumed to be greater than 1.

In the case $(R-1) \gg u$, $a(n+1) \simeq [1 - uR/(R-1)]a(n)$ from Eq. (24). Accordingly points escape the basin bordered

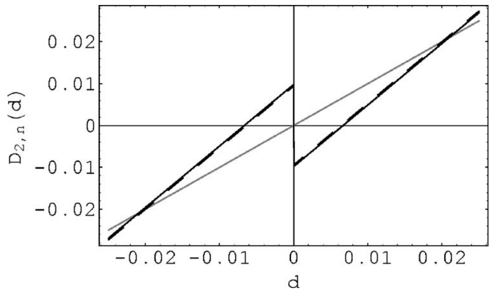

FIG. 8. Applications $D_{2,n}$ given by Eq. (23) with $u=0.02$ and $R=2$. $n=0$ (plain line) and $n=1$ (dashed line).

by $\pm d_{n,F-}$ from time n to $n+1$, if $|d(n+1)|>d_{n+1,F-}\simeq a(n+1)$ implying $|d(n)|>[1-u3R/2(R-1)]a(n)$ (see Fig. 8). Thus, if d belongs to the interval $\Omega_{l,n}$ at time n, successive iterated points will be contained into the successive intervals $\Omega_{l,n}$ (see Fig. 9).

Even if successive iterated points do not escape the basin bordered by $\pm d_{n,F-}$, the dynamics is no longer chaotic. Indeed the point $d=0$ is now asymptotically stable. This is due to the nonstationary dynamics effects. Nevertheless, the shrinking dynamics does not affect the sign of the iterated d. So the expected winner at the issue of an electoral campaign remains unpredictable.

Including unsettled people driven gradually in the public debate on a contrarian basis provides a shrinking of the chaotic basin of the one-sided contrarians model. Thus, the results may shed light on recent very unusual elections like the hung 2000 U.S. presidential and the German 2005 elections. Indeed, the winners of these long electoral campaigns were very unpredictable and the outcomes very tied.

VIII. CONCLUSION

We have presented a simple model giving a deterministic opinion dynamics [see Eq. (9)], which can be chaotic. Furthermore, chaotic basins can coexist with two point attractors at the extremes. This model contains two main effects. First, amplification is given by the local majority rule inside groups and second, retroaction is given by the action of the one-sided contrarians. The one-sided contrarians act by comparison and opposition to a collective information, the majority. Their action introduces into the dynamics a discontinuity.

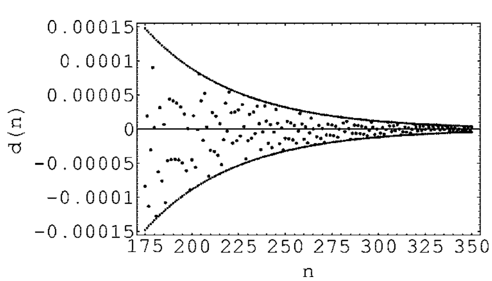

FIG. 9. Successive iterated points with $D_{2,n}$ given by Eq. (23) where $u=0.02$ and $R=2$. Borders $\pm a(n)/2$ are included in thin points. Notation here: $d(n+1)=D_{2,n}[d(n)]$.

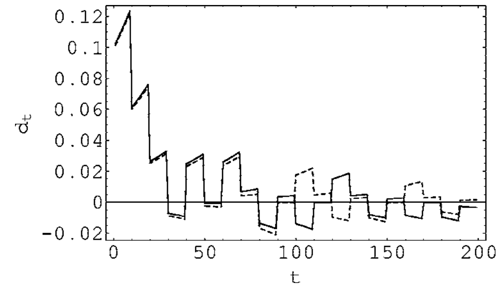

FIG. 10. Combination of a k-step process and unsettled agents with $k=10$, $w=0.05$, $u=0.05$, and $R=1.5$. Initial values are 0.1 (plain line) and (0.1−0.001) (dashed line).

Afterward, rooted in this simple model, some other features of electoral campaigns are added, like the fact that polls are not made public every day, or a bias in favor of one opinion, or the influence of unsettled people gradually driven in the public debate on a contrarian basis.

To sum up our various results we come back to the one-sided contrarian model deterministic equation used throughout this paper:

$$f(x) = (1-a)\left(\frac{3}{2}x - 2x^3\right) - \frac{a}{2}\,\mathrm{sgn}[x], \qquad (25)$$

where $x\equiv d\in[-1/2;1/2]$ and a is the density of one-sided contrarians, $0\leq a\leq 1$. At the second order in x approximation, with $\lambda_m=\frac{3}{2}(1-a)$, $f(x)\simeq\lambda_m x-(a/2)\mathrm{sgn}[x]$. λ_m is the expansion effect. The term $-(a/2)\mathrm{sgn}[x]$ is related to the folding up, i.e., the retroaction effect. The term $-x^3$ can be seen as a nonlinear saturation effect and gives the point attractors. It is symmetric with respect to $x=0$.

Afterward, accounting for some other social parameters enriches the model and modifies somewhat its applications. By decoupling the local update time sequence from the one-sided contrarian behavior activation, the two effects, amplification and folding-up effects, can act separately. Next, a bias in favor of one opinion introduced as a simple parameter s fixes the discontinuity position at $x=s$, i.e., $\mathrm{sgn}[x]\to\mathrm{sgn}[x-s]$. The opinion dynamics is no longer symmetric. Last, unsettled people driven gradually to public debate on a contrarian basis interact with the folding-up effect almost uniquely; $\lambda_m\simeq\frac{3}{2}$ and $a\to a(n)$ with $a(n)\to 0$ when $n\to\infty$. The opinion-forming dynamics is no longer stationary.

At this stage it is worth stressing that, contrary to what could be expected, our treatment using probabilities, local updates, and reshuffling between updates does not define a mean-field-like frame. This result was demonstrated recently using Monte Carlo simulations of a nearest neighbor ferromagnetic Ising system on a square lattice [22]. Indeed it creates a new class of universality in addition to both the two-dimensional Ising and mean-field ones. For our current model, in the case of group size four with no contrarians, using a cellular automaton was shown to recover our analytical result [12].

To conclude it is worth noticing that in terms of real life situations, due to the finite number of iterations imposed by the fact that any public campaign is finite in time, the intention vote dynamics will not exhibit a chaotic behavior although in principle it could. In addition, the nonzero fuzziness of poll measurements of the initial intended vote distribution results automatically in a growing error making it difficult to predict the sign of successive iterations (see Fig. 10).

Although this simple model does not pretend to give an exhaustive explanation of opinion forming during electoral campaigns it exhibits some features that could shed new light on recent surprising voting outcomes, like for instance, on the one hand, an unpredictable issue with a very tied outcome like the German 2005 and the U.S. 2000 elections, and on the other hand, a well-predicted outcome with a huge majority as in the 2002 French presidential elections with a majority around 80%.

However, it is worth stressing that at this stage if we are able to embody the above contradictory voting outcomes within a single frame, we are not in a position to select the prevailing one. To overpass this difficulty would require collaboration with social scientists to estimate the actual type of contrarians involved in a given election. It is open for future work.

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